

# ORDINARY DIFFERENTIAL SYSTEM IN DIMENSION SIX WITH AFFINE WEYL GROUP SYMMETRY OF TYPE $D_4^{(2)}$

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**ABSTRACT.** We find a three-parameter family of ordinary differential systems in dimension six with affine Weyl group symmetry of type  $D_4^{(2)}$ . This is the second example which gave higher order Painlevé type systems of type  $D_4^{(2)}$ . We show that we give its symmetry and holomorphy conditions. These symmetries, holomorphy conditions and invariant divisors are new.

## 1. INTRODUCTION

In [19], we study a two-parameter (resp. four-parameter) family of ordinary differential systems with affine Weyl group symmetry of type  $D_3^{(2)}$  (resp.  $D_5^{(2)}$ ). They are considered to be higher order versions of  $P_{II}$ . These systems are equivalent to the polynomial Hamiltonian systems, and can be considered to be 2-coupled (resp. 4-coupled) Painlevé II systems in dimension four (resp. eight).

We will complete the study of the above problem in a series of papers, for which this paper is the second, resulting in a series of equations for the remaining affine root systems of types  $D_l^{(2)}$  ( $l = 4, 6, 7, \dots$ ). This paper is the stage in this project where we find a 3-parameter family of ordinary differential systems in dimension six with affine Weyl group symmetry of type  $D_4^{(2)}$  given by

$$(1) \quad \frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

with the polynomial Hamiltonian

$$(2) \quad \begin{aligned} H = & \frac{1}{4t}y^3 + \frac{3}{2}y^2 + \frac{3\alpha_3 - 1}{t}xy + \frac{3}{4t}z^2w^2 + \frac{3}{2}z^2w + \frac{3\alpha_1 + 3\alpha_2 - 2}{2t}zw + \frac{3}{2}\alpha_1z \\ & - \frac{4}{t}p^3 - 6p^2 - \frac{3\alpha_1 + 3\alpha_2 + 3\alpha_3 - 2}{t}qp - 6tp + \frac{3}{4t}\alpha_1(8xp + 2zp + yz) + \frac{6}{t}\alpha_2xp \\ & + \frac{3}{2t}\alpha_3(4xp - yq) + \frac{3}{4t}(8xyqp - 4zwqp + 8xzw p - 8x^2yp + 2z^2wp + yz^2w \\ & - 2yq^2p + 8w^2p - 4yp^2 + 4yw^2 + 4y^2w + 8ywp - 8xp + 8tyw). \end{aligned}$$

Here  $x, y, z, w, q$  and  $p$  denote unknown complex variables, and  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  are complex parameters satisfying the relation:

$$(3) \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

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In section 2, each principal part of this Hamiltonian can be transformed into the one with its first integrals by birational and symplectic transformations. However, the Hamiltonian  $H$  is not the first integral.

We remark that for this system we tried to seek its first integrals of polynomial type with respect to  $x, y, z, w, q, p$ . However, we can not find.

This is the second example which gave higher order Painlevé type systems of type  $D_4^{(2)}$ .

We also remark that 2-coupled Painlevé III system in dimension four given in the paper [11] admits the affine Weyl group symmetry of type  $D_4^{(2)}$  as the group of its Bäcklund transformations, whose generators  $w_1, w_2$  are determined by the invariant divisors. However, the transformations  $w_3, w_4$  do not satisfy so (see Theorem 4.1 in [11]).

On the other hand, the system (1) admits the affine Weyl group symmetry of type  $D_4^{(2)}$  as the group of its Bäcklund transformations, whose generators  $s_0, \dots, s_3$  are determined by the invariant divisors (3.2).

## 2. PRINCIPAL PARTS OF THE HAMILTONIAN

In this section, we study three Hamiltonians  $K_1, K_2$  and  $K_3$  in the Hamiltonian  $H$ .

At first, we study the Hamiltonian system

$$(4) \quad \begin{aligned} \frac{dx}{dt} &= \frac{\partial K_1}{\partial y} = \frac{3y(y+4t) - 4(\alpha_0 + \alpha_1 + \alpha_2 - 2\alpha_3)x}{4t}, \\ \frac{dy}{dt} &= -\frac{\partial K_1}{\partial x} = \frac{(\alpha_0 + \alpha_1 + \alpha_2 - 2\alpha_3)y}{t} \end{aligned}$$

with the polynomial Hamiltonian

$$(5) \quad K_1 = \frac{1}{4t}y^3 + \frac{3}{2}y^2 + \frac{3\alpha_3 - 1}{t}xy,$$

where setting  $z = w = q = p = 0$  in the Hamiltonian  $H$ , we obtain  $K_1$ .

This equation can be explicitly solved by

$$(6) \quad \begin{aligned} x(t) &= \frac{C_1 t^{-1+3(\alpha_0+\alpha_1+\alpha_2)}}{\alpha_0 + \alpha_1 + \alpha_2 - \alpha_3} + \frac{C_1^2 t^{-4+6(\alpha_0+\alpha_1+\alpha_2)}}{4(\alpha_0 + \alpha_1 + \alpha_2 - 2\alpha_3)} + C_2 t^{2-3(\alpha_0+\alpha_1+\alpha_2)}, \\ y(t) &= C_1 t^{(\alpha_0+\alpha_1+\alpha_2-2\alpha_3)} \quad (C_1, C_2 : \text{integral constants}). \end{aligned}$$

Next, we study the Hamiltonian system

$$(7) \quad \frac{dz}{dt} = \frac{\partial K_2}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial K_2}{\partial z}$$

with the polynomial Hamiltonian

$$(8) \quad K_2 = \frac{3}{4t}z^2w^2 + \frac{3}{2}z^2w + \frac{3\alpha_1 + 3\alpha_2 - 2}{2t}zw + \frac{3}{2}\alpha_1 z,$$

where setting  $x = y = q = p = 0$  in the Hamiltonian  $H$ , we obtain  $K_2$ .

**Step 1:** We make the change of variables:

$$(9) \quad z_1 = tz, \quad w_1 = \frac{w}{t}.$$

We remark that this transformation is symplectic.

It is easy to see that the system with the polynomial Hamiltonian

$$(10) \quad \tilde{K}_2 = \frac{3z_1(z_1w_1^2 + 2z_1w_1 + 2(\alpha_1 + \alpha_2)w_1 + 2\alpha_1)}{4t}$$

has its first integral I:

$$(11) \quad I := 4t\tilde{K}_2.$$

Finally, we study the Hamiltonian system

$$(12) \quad \begin{aligned} \frac{dq}{dt} &= \frac{\partial K_3}{\partial p} = -\frac{12p(p+t) - (2\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3)q + 6t^2}{t}, \\ \frac{dp}{dt} &= -\frac{\partial K_3}{\partial q} = -\frac{(2\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3)p}{t} \end{aligned}$$

with the polynomial Hamiltonian

$$(13) \quad K_3 = -\frac{4}{t}p^3 - 6p^2 - \frac{3\alpha_1 + 3\alpha_2 + 3\alpha_3 - 2}{t}qp - 6tp,$$

where setting  $x = y = z = w = 0$  in the Hamiltonian  $H$ , we obtain  $K_3$ .

This equation can be explicitly solved by

$$(14) \quad \begin{aligned} q(t) &= -2t^2 \left\{ \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} + 2C_1 t^{-6\alpha_0} \left( \frac{t^{3\alpha_0}}{-\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3} + \frac{C_1}{-2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3} \right) \right\} \\ &\quad + C_2 t^{-1+3\alpha_0}, \\ p(t) &= C_1 t^{-(2\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3)} \quad (C_1, C_2 : \text{integral constants}). \end{aligned}$$

### 3. SYMMETRY AND HOLOMORPHY CONDITIONS

In this section, we study the symmetry and holomorphy conditions of the system (1). These properties are new.

**THEOREM 3.1.** *The system (1) admits the affine Weyl group symmetry of type  $D_4^{(2)}$  as the group of its Bäcklund transformations, whose generators  $s_0, s_1, \dots, s_3$  defined as follows: with the notation  $(*) := (x, y, z, w, q, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3)$ :*

$$(15) \quad \begin{aligned} s_0 : (*) &\rightarrow \left( x + \frac{\alpha_0}{y + z^2/4}, y, z, w - \frac{\alpha_0 z}{2(y + z^2/4)}, q, p, t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3 \right), \\ s_1 : (*) &\rightarrow \left( x, y, z + \frac{\alpha_1}{w}, w, q, p, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3 \right), \\ s_2 : (*) &\rightarrow \left( x + \frac{\alpha_2/2}{f_2}, y - \frac{\alpha_2 z/2}{f_2} - \frac{\alpha_2^2/4}{f_2^2}, z + \frac{\alpha_2}{f_2}, w + \frac{\alpha_2(q - 2x)/4}{f_2}, \right. \\ &\quad \left. q + \frac{\alpha_2}{f_2}, p + \frac{\alpha_2 z/4}{f_2} + \frac{\alpha_2^2/8}{f_2^2}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2 \right), \\ s_3 : (*) &\rightarrow \left( x, y, z, w, q + \frac{\alpha_3}{p}, p, t; \alpha_0, \alpha_1, \alpha_2 + 2\alpha_3, -\alpha_3 \right), \end{aligned}$$

where  $f_2 := w + p + \frac{y}{2} + \frac{xz}{2} - \frac{zq}{4} + t$ .

We note that the Bäcklund transformations of this system satisfy

$$(16) \quad s_i(g) = g + \frac{\alpha_i}{f_i} \{f_i, g\} + \frac{1}{2!} \left( \frac{\alpha_i}{f_i} \right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[x, y, z, w, q, p]),$$

where poisson bracket  $\{, \}$  satisfies the relations:

$$\{y, x\} = \{w, z\} = \{p, q\} = 1, \quad \text{the others are 0.}$$

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

PROPOSITION 3.2. *This system has the following invariant divisors:*

parameter's relation	$f_i$
$\alpha_0 = 0$	$f_0 := y + \frac{z^2}{4}$
$\alpha_1 = 0$	$f_1 := w$
$\alpha_2 = 0$	$f_2 := w + p + \frac{y}{2} + \frac{xz}{2} - \frac{zq}{4} + t$
$\alpha_3 = 0$	$f_3 := p$

We note that when  $\alpha_1 = 0$ , we see that the system (1) admits a particular solution  $w = 0$ , and when  $\alpha_2 = 0$ , after we make the birational and symplectic transformations:

$$(17) \quad x_2 = x - \frac{z}{2}, \quad y_2 = y + \frac{z^2}{4}, \quad z_2 = z, \quad w_2 = w + \frac{y}{2} + p + t + \frac{xz}{2} - \frac{zq}{4}, \quad q_2 = q - z, \quad p_2 = p - \frac{z^2}{8}.$$

we see that the system (1) admits a particular solution  $w_2 = 0$ .

PROPOSITION 3.3. *Let us define the following translation operators:*

$$(18) \quad T_1 := s_1 s_2 s_3 s_2 s_1 s_0, \quad T_2 := s_1 T_1 s_1, \quad T_3 := s_2 T_2 s_2.$$

*These translation operators act on parameters  $\alpha_i$  as follows:*

$$(19) \quad \begin{aligned} T_1(\alpha_0, \alpha_1, \alpha_2, \alpha_3) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3) + (-2, 2, 0, 0), \\ T_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3) + (0, -2, 2, 0), \\ T_3(\alpha_0, \alpha_1, \alpha_2, \alpha_3) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3) + (0, 0, -2, 2). \end{aligned}$$

THEOREM 3.4. *Let us consider a polynomial Hamiltonian system with Hamiltonian  $K \in \mathbb{C}(t)[x, y, z, w, q, p]$ . We assume that*

(A1)  *$\deg(K) = 4$  with respect to  $x, y, z, w, q, p$ .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate system  $r_i$  ( $i = 0, 1, 2, 3$ ):*

(20)

$$\begin{aligned} r_0 : x_0 &= \frac{1}{x}, \quad y_0 = -\left(\left(y + \frac{z^2}{4}\right)x + \alpha_0\right)x, \quad z_0 = z, \quad w_0 = w + \frac{xz}{2}, \quad q_0 = q, \quad p_0 = p, \\ r_1 : x_1 &= x, \quad y_1 = y, \quad z_1 = \frac{1}{z}, \quad w_1 = -(wz + \alpha_1)z, \quad q_1 = q, \quad p_1 = p, \\ r_2 : x_2 &= x - \frac{z}{2}, \quad y_2 = y + \frac{z^2}{4}, \quad z_2 = \frac{1}{z}, \quad w_2 = -\left(\left(w + \frac{y}{2} + p + t + \frac{xz}{2} - \frac{zq}{4}\right)z + \alpha_2\right)z, \\ q_2 &= q - z, \quad p_2 = p - \frac{z^2}{8}, \\ r_3 : x_3 &= x, \quad y_3 = y, \quad z_3 = z, \quad w_3 = w, \quad q_3 = \frac{1}{q}, \quad p_3 = -(pq + \alpha_3)q. \end{aligned}$$

*Then such a system coincides with the system*

$$(21) \quad \frac{dx}{dt} = \frac{\partial K}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial K}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial K}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial K}{\partial z}, \quad \frac{dq}{dt} = \frac{\partial K}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial K}{\partial q}$$

*with the polynomial Hamiltonian*

$$(22) \quad K = H + a_1(y + 2p) + a_2(y + 2p)^2 + a_3(y + 2p)^3 + a_4(y + 2p)^4 \quad (a_i \in \mathbb{C}(t)).$$

We note that the condition (A2) should be read that

$$r_j(K) \quad (j = 0, 1, 3), \quad r_2(K - z)$$

are polynomials with respect to  $x_i, y_i, z_i, w_i, q_i, p_i$ .

We remark that  $y + 2p$  is not the first integral of the system (21) with the polynomial Hamiltonian (22).

## REFERENCES

- [1] P. Painlevé, *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme*, Bull. Société Mathématique de France. **28** (1900), 201–261.
- [2] P. Painlevé, *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale est uniforme*, Acta Math. **25** (1902), 1–85.
- [3] B. Gambier, *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes*, Acta Math. **33** (1910), 1–55.
- [4] C. M. Cosgrove and G. Scoufis, *Painlevé classification of a class of differential equations of the second order and second degree*, Studies in Applied Mathematics. **88** (1993), 25–87.
- [5] C. M. Cosgrove, *All binomial-type Painlevé equations of the second order and degree three or higher*, Studies in Applied Mathematics. **90** (1993), 119–187.
- [6] F. Bureau, *Integration of some nonlinear systems of ordinary differential equations*, Annali di Matematica. **94** (1972), 345–359.
- [7] J. Chazy, *Sur les équations différentielles dont l'intégrale générale est uniforme et admet des singularités essentielles mobiles*, Comptes Rendus de l'Académie des Sciences, Paris. **149** (1909), 563–565.

- [8] J. Chazy, *Sur les équations différentielles dont l'intégrale générale possède une coupure essentielle mobile*, Comptes Rendus de l'Académie des Sciences, Paris. **150** (1910), 456–458.
- [9] J. Chazy, *Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale a ses points critiques fixes*, Acta Math. **34** (1911), 317–385.
- [10] Y. Sasano, *Coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of types  $B_6^{(1)}$ ,  $D_6^{(1)}$  and  $D_7^{(2)}$* , preprint.
- [11] Y. Sasano, *Four-dimensional Painlevé systems of types  $D_5^{(1)}$  and  $B_4^{(1)}$* , preprint.
- [12] Y. Sasano, *Higher order Painlevé equations of type  $D_l^{(1)}$* , RIMS Kokyuroku **1473** (2006), 143–163.
- [13] Y. Sasano, *Symmetries in the system of type  $D_4^{(1)}$* , preprint.
- [14] Y. Sasano, *Coupled Painlevé III systems with affine Weyl group symmetry of types  $B_4^{(1)}$ ,  $D_4^{(1)}$  and  $D_5^{(2)}$* , preprint.
- [15] Y. Sasano, *Coupled Painlevé III systems with affine Weyl group symmetry of types  $B_5^{(1)}$ ,  $D_5^{(1)}$  and  $D_6^{(2)}$* , preprint.
- [16] Y. Sasano, *Coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type  $D_6^{(1)}$ , II*, RIMS Kokyuroku Bessatsu. **B5** (2008), 137–152.
- [17] Y. Sasano, *Coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type  $E_6^{(2)}$* , preprint.
- [18] Y. Sasano, *Symmetry in the Painlevé systems and their extensions to four-dimensional systems*, Funkcial. Ekvac. **51** (2008), 351–369.
- [19] Y. Sasano, *Symmetries in the systems of types  $D_3^{(2)}$  and  $D_5^{(2)}$* , preprint.